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RAPID MODE-III CRACK PROPAGATION IN A STRIP OF VISCOPLASTIC WORK-HARDENING MATERIAL

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Abstract—Steady-state dynamic fields of stress and deformation have been computed for rapid crack propagation in the mid-plane of a thick strip of a viscoplastic work-hardening material. The faces of the strip are subjected to prescribed anti-plane displacements. The system of governing equations has been solved numerically by a finite difference method with an iterative procedure in the time domain which is continued until a steady-state has been reached. The effects of viscoplastic material behavior have been exhibited by comparisons with the results for the corresponding perfectly elastic problem. Dynamic effects due to high crack-tip speeds have been investigated and dynamic results have been compared with results obtained on the basis of a quasi-static problem formulation.

INTRODUCTION

The fields of stress and deformation near the tip of a stationary crack in elastic–plastic rate-independent materials have been studied in some detail, see[1], for examples and references. The problem of a growing crack in inelastic materials is more difficult, and only a few continuum plasticity solutions are available, see[2] for a review. The main difficulty stems from the history-dependent nature of the constitutive equations and the feature that the material experiences a nonproportional straining history when it is loaded and unloaded as the crack passes by.

Most of the existing treatments deal with quasi-static crack propagation in elastic–plastic materials under a small scale yielding assumption in which the zone of plastic deformation is assumed to be sufficiently localized near the crack tip that the singular part of the elastic solution dominates the elastic field near the yield zone. For a recent work on quasi-static steady crack growth under small scale yielding conditions see Dean and Hutchinson[3].

The literature on dynamic effects in the presence of elastic–plastic constitutive behaviour is very limited. Investigations of the dynamic near-tip fields in an elastic perfectly plastic material were presented by Slepyan[4] and Achenbach and Dunayevsky[5]. Dynamic near-tip effects for the case of a strain-hardening material were investigated by Achenbach and Kanninen[6] and Achenbach *et al.*[7], on the basis of J_2 flow theory and a bilinear effective stress–strain relation. These authors found results which are very similar to the ones obtained by Amazigo and Hutchinson[8] for the corresponding quasi-static problem.

In this paper we investigate both the effects of plastic deformation near a propagating crack tip and dynamic effects due to high crack-tip speeds. The constitutive equations that are employed define an elastic viscoplastic material. The constitutive model does not require the statement of a separate yield criterion, nor is it necessary to consider loading and unloading separately. Plastic deformations always exist, but they are negligibly small when the material behavior should be essentially elastic.

The geometry that is considered is a two-dimensional one of a thick strip which contains a rapidly propagating semi-infinite crack in its center-plane. The faces of the strip are subjected to uniform anti-plane displacements, so that the crack propagates in Mode-III. A steady-state situation relative to the moving crack tip has been assumed. The plastic deformations near the crack tip, the residual plastic strains in the wake of the crack tip and other field variables are obtained directly from the complete solution without the assumption of small scale yielding.

The method of solution is numerical and involves an iterative procedure which is continued until a steady state solution has been reached, in which the equation of motion, the flow rules

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and boundary conditions are satisfied simultaneously. The method is based on a finite difference procedure which is unconditionally stable and has a second order accuracy.

In the special case of a perfectly elastic strip the problem possesses an analytical solution[9–10] which has been employed to check the accuracy of the numerical method, and excellent agreement has been obtained. For a comprehensive review on various dynamic crack problems in perfectly elastic strips, see the review paper by Nilsson[11].

Typical effects due to viscoplastic consititutive behavior are studied by comparisons with the corresponding elastic fields. The effects of high crack-tip speeds, which are directly related to the strain-rate dependence of the material, are studied by comparisons of solutions for three crack-tip velocities. The influence of the inertia term in the governing equations is studied by comparisons with the corresponding quasi-static solutions. In particular, the dependence on the crack-tip speed of the plastic zone in the vicinity of the crack tip, the level of plastic straining, the amount of dissipative plastic work and the opening displacements are shown. A secondary plastic loading zone within the wake region is observed for the dynamic problems at high crack-tip speeds. This region appears to be absent in the quasi-static problems.

2. CONSTITUTIVE EQUATIONS

A convenient set of constitutive equations for an elastic-viscoplastic work-hardening material has been proposed by Bodner and Partom[12]. These equations have the useful property that no separate specification of a yield criterion is required, nor is it necessary to consider loading and unloading separately. Within the context of these equations both elastic and inelastic deformations are present at all stages of loading and unloading, but the plastic deformations are very small when the material behavior should be essentially elastic.

In the usual manner the total rate of strain is expressed as the superposition of elastic (reversible) and plastic (irreversible) components:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(e)} + \dot{\epsilon}_{ij}^{(p)} \qquad i, j = 1,2,3. \tag{1}$$

Here ϵ_{ij} are the components of the infinitesimal strain tensor, $\epsilon_{ij} = 1/2(u_{i,j} + u_{j,i})$, where u_i are the components of the displacement vector, and dots represent time derivatives. The elastic strain rates are related to the stress-rates, $\dot{\sigma}_{ij}$, by Hooke's law

$$\dot{\epsilon}_{ij}^{(e)} = \frac{1}{2\mu}\left(\dot{\sigma}_{ij} - \frac{\nu}{1-\nu}\,\dot{\sigma}_{kk}\,\delta_{ij}\right) \tag{2}$$

where μ is the shear modulus, ν is Poisson's ratio and δ_{ij} is the Kronecker delta. It is assumed that the plastic deformations are incompressible ($\dot{\epsilon}_{kk}^{(p)} = 0$) and that the Prandtl–Reuss flow law holds. Thus

$$\dot{\epsilon}_{ij}^{(p)} = \dot{e}_{ij}^{(p)} = \Lambda\, s_{ij} \tag{3}$$

where s_{ij} and $\dot{e}_{ij}^{(p)}$ denote the deviators of the stress tensor and the plastic strain-rate tensor, respectively, i.e. $s_{ij} = \sigma_{ij} - 1/3\,\sigma_{kk}\,\delta_{ij}$ and $\dot{e}_{ij}^{(p)} = \dot{\epsilon}_{ij}^{(p)} - 1/3\,\dot{\epsilon}_{kk}^{(p)}\,\delta_{ij}$. Equation (3) can be squared to yield Λ in the form

$$\Lambda^2 = D_2^{(p)}/J_2. \tag{4}$$

Here

$$D_2^{(p)} = \tfrac{1}{2}\,\dot{e}_{ij}^{(p)}\,\dot{e}_{ij}^{(p)}, \text{ and } J_2 = \tfrac{1}{2}\,s_{ij}\,s_{ij} \tag{5a, b}$$

are the second invariants of the plastic strain-rate deviator and the stress deviator, respectively.

Motivated by equations relating dislocation velocities and stresses, Bodner and Partom[12] proposed the relation

$$D_2^{(p)} = D_0^2 \exp\left[-(A^2/J_2)^n\right], \tag{6}$$

where

$$A^2 = \tfrac{1}{3} Z^2[(n+1)/n]^{1/n}. \tag{7}$$

The coefficient n is related to the steepness of the $D_2^{(p)} - J_2$ curve, D_0^2 is the limiting value of $D_2^{(p)}$ for very high stresses and Z is an internal state variable referred to as the hardness of the material, which expresses its overall resistance to plastic flow. For isotropic work-hardening the evolution equation for Z is taken to depend on the amount of plastic (irreversible) work, W_p, which has been done on the material from a reference state. Specifically, Z is assumed to have the form

$$Z = Z_1 + (Z_0 - Z_1)\exp[-m\, W_p/Z_0] \tag{8}$$

where Z_0, Z_1 and m are appropriate paramters of the material and the rate of plastic work can be expressed in the form

$$\dot{W}_p = \sigma_{ij}\,\dot{\epsilon}_{ij}^{(p)} = s_{ij}\,\dot{\epsilon}_{ij}^{(p)} = 2\Lambda\, J_2. \tag{9}$$

In eqn (8), Z_0 is the initial hardness and Z_1 is the upper limit of Z (saturation value). The hardness must have an upper limit, because other wise $D_2^{(p)}$ would approach zero for large W_p, which would imply fully elastic behavior at appreciable strains.

It can be shown that for the case of uniaxial stress under a prescribed strain rate, the axial stress asymptotically approaches a maximum value as the total strain increases. The limiting stress becomes larger as the applied strain rate increases.

The system of governing equations is completed with the stress-equations of motion

$$\sigma_{ij,j} = \rho\, \ddot{u}_i \tag{10}$$

where ρ is the mass density.

3. A STEADILY MOVING CRACK IN A STRIP

Let us consider the anti-plane deformations of a strip of height $2h$ whose constitutive behavior is defined by eqns (1)–(9). The one non-vanishing displacement component is $u_3(x_1,x_2,t)$ and the corresponding total strains are

$$\epsilon_{31} = \tfrac{1}{2}\, u_{3,1}; \quad \epsilon_{32} = \tfrac{1}{2}\, u_{3,2}. \tag{11a, b}$$

The non-zero stress components follow from (1) and (2) as

$$\sigma_{31} = 2\mu(\epsilon_{31} - \epsilon_{31}^{(p)}); \quad \sigma_{32} = 2\mu(\epsilon_{32} - \epsilon_{32}^{(p)}). \tag{12a, b}$$

The second invariant of the stress deviator follows from (5b) as

$$J_2 = \sigma_{31}^2 + \sigma_{32}^2. \tag{13}$$

For anti-plane strain the equation of motion reduces to

$$\mu(u_{3,11} + u_{3,22}) - 2\mu\,(\epsilon_{31,1}^{(p)} + \epsilon_{32,2}^{(p)}) = \rho\ddot{u}_3. \tag{14}$$

The plastic strains are governed by the flow rule (3).

As loading of the strip we assume a uniform displacement of the faces at $x_2 = \pm h$ over a distance $\pm w_0$.

Let us first consider the quasi-static states of stress and deformation in a strip which does not contain a crack. By virtue of anti-symmetry it suffices to consider the upper half of the strip, with the following boundary conditions

$$u_3(x_1,h,t) = w_0 H(t) \tag{15}$$

and

$$u_3(x_1,0,t) = 0 \tag{16}$$

where $H(t)$ is the Heaviside step function. At time $t = 0$ the strip is free of deformation.

Since the field variables are obviously independent of x_1 the quasi-static equilibrium equation can be solved to yield $\sigma_{32} = \sigma_{32}(t)$. The homogeneity of the strip implies that the relevant total strain ϵ_{32} is simply $\epsilon_{32} = w_0/2h$. Hence eqn (12) yields

$$\sigma_{32}(t) = \mu[w_0/h - 2\epsilon_{32}^{(p)}(t)]. \tag{17}$$

The plastic strain and the plastic work, which are governecd by (3) and (9), respectively, take the form

$$\dot{\epsilon}_{32}^{(p)} = \Lambda(t)\,\sigma_{32}(t) \tag{18}$$

$$\dot{W}_p(t) = 2\Lambda(t)\,\sigma_{32}^2(t). \tag{19}$$

The system of nonlinear equations (17)–(19) can be easily integrated numerically, to yield the asymptotic values for σ_{32}, $\epsilon_{32}^{(p)}$ and W_p, which are reached at long times.

Next we consider the case that the center-plane of the strip contains a semi-infinite crack which propagates in the x_1-direction with a constant velocity v. The geometry is shown in Fig. 1. The (x_1,x_2,x_3) coordinate is now assumed to move with the crack tip. A stationary coordinate system (x,y,z) and the moving coordinates (x_1,x_2,x_3) are related by

$$x_1 = x - vt,\ x_2 = y,\ x_3 = z. \tag{20}$$

It is further assumed that the crack has been moving for a long time and that a steady-state has been established relative to the moving coordinate system. The time derivatives in the equation of motion then reduce to

$$(\dot{\ }) = -v\frac{\partial}{\partial x_1}, (\ddot{\ }) = v^2\frac{\partial^2}{\partial x_1^2}. \tag{21}$$

Consequently the equation of motion reduces to

$$(1 - v^2/c^2)\,u_{3,11} + u_{3,22} - 2(\epsilon_{31,1}^{(p)} + \epsilon_{32,2}^{(p)}) = 0 \tag{22}$$

where $c^2 = \mu/\rho$. In the moving coordinate system the flow equations become

$$v\,\epsilon_{31,1}^{(p)} + \Lambda\,\sigma_{31} = 0 \tag{23}$$

$$v\,\epsilon_{32,1}^{(p)} + \Lambda\,\sigma_{32} = 0 \tag{24}$$

and expression (9) for the rate of plastic work reduces to

$$v(W_p)_{,1} + 2\Lambda J_2 = 0. \tag{25}$$

Equations (22)–(25) form a system of nonlinear differential equations for the field variables $u_3(x_1, x_2)$, $\epsilon_{31}^{(p)}(x_1,x_2)$, $\epsilon_{32}^{(p)}(x_1,x_2)$, $W_p(x_1,x_2)$. By rewriting these equations as a system of five first order differential equations in ϵ_{31}, ϵ_{32}, $\epsilon_{31}^{(p)}$, $\epsilon_{32}^{(p)}$, W_p and by using the compatibility condition for the total strains, it can be shown that the system is of elliptic type as long as $v/c < 1$, i.e. for subsonic crack growth.

The boundary condition on the surface of the crack is

$$\sigma_{32} = 0 \ \text{for}\ -\infty < x_1 \le 0 \tag{26}$$

while the condition of displacement antisymmetry yields

$$u_3(x_1,0) = 0 \ \text{for}\ x_1 > 0. \tag{27}$$

At some distance ahead of the crack tip the fields are equal to the long-time solutions for the uncracked strip, discussed earlier in this section.

For a perfectly elastic strip the problem possesses an analytical solution given by Field and Baker[9] and by Sih and Chen[10].

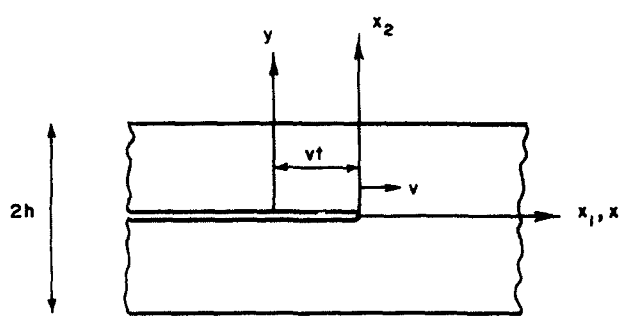

Fig. 1. Propagating crack in a strip.

4. NUMERICAL TREATMENT

The system of eqns (22)–(25) has been solved numerically by a finite difference procedure which employed a grid of mesh sizes Δx_1 and Δx_2 in the x_1 and x_2 directions, respectively. The procedure can be divided into two steps. In the first step (22) is treated, while (23)–(25) are integrated in the second step.

Instead of solving directly the finite difference approximations to the elliptic equations, which would involve a large system of non-linear algebraic equations, we adopt a different approach in which the following equation is considered in place of (22):

$$a\frac{\partial u_3}{\partial t} = (1 - v^2/c^2)\, u_{3,11} + u_{3,22} - 2\,[\epsilon_{31,1}^{(p)} + \epsilon_{32,2}^{(p)}]. \tag{28}$$

Here a is an appropriate coefficient which has the dimension of $(\text{length}^2/\text{time})^{-1}$. Equation (28) is a parabolic equation governing $u_3 = u_3(x_1,x_2,t)$. The steady-state solution to this equation and eqns (23)–(25), which satisfies the boundary conditions (26)–(27) and conditions at large $|x_1|$, forms the desired solution to the steady-state fields generated by a propagating crack in the strip.

Equation (28) is solved by the generalized Du Fort-Frankel method for parabolic initial-boundary value problems, proposed recently by Gottlieb and Gustafsson[13] and applied by these authors to hydrodynamic equations. The steady-state solution is obtained after long enough time. If Δt defines a time increment, this scheme yields an explicit three-level procedure according to which it is possible to compute u_3 at internal points of the strip at time $t + \Delta t$ whenever its values at the two previous time levels as well as the plastic strains at time t are known throughout the strip. The advantage of this method is that is does not involve the solution of large systems of non-linear algebraic equations and since it is explicit and unconditionally stable it produces the solution rather easily and efficiently. The steady state solution is achieved very rapidly, the condition being that the solution does not change further with time with a preassigned degree of accuracy.

A scheme of second-order accuracy has been adopted by which the error resulting from the replacement of the differential equation (28) by its difference approximation is of second order in the increments. With $\Delta x_1 = \Delta x_2$, which is the usual case in practical computations, the explicit form of the difference approximation to (28) can be written in the form:

$$\begin{aligned} e_2\, u_3(x_1,x_2,t+\Delta t) &= e_3 u_3(x_1,x_2,t-\Delta t) + 2(\gamma-1)e_1\, u_3(x_1,x_2,t) \\ &\quad + (1 - v^2/c^2)\,[u_3(x_1+\Delta x_1,x_2,t) + u_3(x_1-\Delta x_1,x_2,t)] \\ &\quad + u_3(x_1,x_2+\Delta x_2,t) + u_3(x_1,x_2-\Delta x_2,t) - \Delta x_1[\epsilon_{31}^{(p)}(x_1+\Delta x_1,x_2,t) \\ &\quad - \epsilon_{31}^{(p)}(x_1-\Delta x_1,x_2,t) + \epsilon_{32}^{(p)}(x_1,x_2+\Delta x_2,t) - \epsilon_{32}^{(p)}(x_1,x_2-\Delta x_2,t)] \end{aligned} \tag{29}$$

where

$$\begin{aligned} e_1 &= 2 - v^2/c^2 \\ e_2 &= \tfrac{1}{2}\epsilon^2 + \gamma e_1 \\ e_3 &= \tfrac{1}{2}\epsilon^2 - \gamma e_1 \\ \epsilon &= a(\Delta x_1)^2/\Delta t \end{aligned}$$

and γ is a parameter to be chosen such that the scheme is unconditionally stable.

The stability of the scheme (29) can be investigated by following the analysis presented in[13]. It is found that for unconditional stability γ must be greater than 1.

In the second part, at every time step at which $u_3(x_1,x_2,t)$ has been computed according to (29), $\epsilon_{31}^{(p)}(x_1,x_2,t), \epsilon_{32}^{(p)}(x_1,x_2,t)$ and $W_p(x_1,x_2,t)$ are computed from (23)–(25) by using the improved Euler–Cauchy method and employing the boundary conditions (26) and (27), together with the limiting values of $\epsilon_{32}^{(p)}$ and W_p at $x_1 > 0$ far away from the tip, which are given by the solution of (18)–(19).

According to this method (23)–(25) are integrated to yield the predicted values

$$\left.\begin{aligned} \overset{*}{\epsilon}{}_{31}^{(p)}(x_1-\Delta x_1,x_2,t) &= \epsilon_{31}^{(p)}(x_1,x_2,t) + \Delta x_1 \Lambda(x_1,x_2,t)\,\sigma_{31}(x_1,x_2,t) \\ \overset{*}{\epsilon}{}_{32}^{(p)}(x_1-\Delta x_1,x_2,t) &= \epsilon_{32}^{(p)}(x_1,x_2,t) + \Delta x_1 \Lambda(x_1,x_2,t)\,\sigma_{32}(x_1x_2,t) \\ \overset{*}{W}_p(x_1-\Delta x_1,x_2,t) &= W_p(x_1,x_2,t) + 2\Delta x_1 \Lambda(x_1,x_2,t)\, J_2(x_1,x_2,t). \end{aligned}\right\} \tag{30}$$

These values are corrected according to

$$\begin{aligned}
\epsilon_{31}^{(p)}(x_1-\Delta x_1,x_2,t) &= \epsilon_{31}^{(p)}(x_1,x_2,t)+\tfrac{1}{2}\Delta x_1[\Lambda(x_1,x_2,t)\,\sigma_{31}(x_1,x_2,t)\\
&\quad+\overset{*}{\Lambda}(x_1-\Delta x_1,x_2,t)\,\overset{*}{\sigma}_{31}(x_1-\Delta x_1,x_2,t)]\\
\epsilon_{32}^{(p)}(x_1-\Delta x_1,x_2,t) &= \epsilon_{32}^{(p)}(x_1,x_2,t)+\tfrac{1}{2}\Delta x_1[\Lambda(x_1,x_2,t)\,\sigma_{32}(x_1,x_2,t)\\
&\quad+\overset{*}{\Lambda}(x_1-\Delta x_1,x_2,t)\,\overset{*}{\sigma}_{32}(x_1-\Delta x_1,x_2,t)]\\
W_p(x_1-\Delta x_1,x_2,t) &= W_p(x_1,x_2,t)+\Delta x_1[\Lambda(x_1,x_2,t)\,J_2(x_1,x_2,t)\\
&\quad+\overset{*}{\Lambda}(x_1-\Delta x_1,x_2,t)\,\overset{*}{J}_2(x_1-\Delta x_1,x_2,t)].
\end{aligned}\tag{31}$$

where the asterisk on Λ and J_2 means that these quantities are evaluated by using the predicted values given by (30).

This process is applied at each time step for $0 \le x_2 \le h$ until the steady state solution has been reached. It turns out that for $x_1/h = 2$, the effect of the crack becomes negligible and the values of $\epsilon_{32}^{(p)}$ and W_p at that position can be taken as the long-time solution of (18)–(19) with u_3 given by w_0x_2/h and $\epsilon_{31}^{(p)} = 0$. Similarly, at the points $x_1 = -2h$, behind the crack tip, the x_1-dependence can be neglected and the value of the field variables can be taken equal to those computed at $x_1 = -2h + \Delta x_1$.

We conclude this section with the observation that no special treatment was given to the computation of the field variables at the tip of the crack or in it's vicinity. The numerical scheme described above has been applied at all points of the strip.

5. RESULTS

The method of solution has been applied to compute the fields generated by crack propagation in a strip made of titanium, for which the material parameters are, see[12]: $\mu = 0.44 \cdot 10^5\ \text{N/mm}^2$, $\rho = 4.87 \cdot 10^3\ \text{Kg/m}^3$, $Z_0 = 1150\ \text{N/mm}^2$, $Z_1 = 1400\ \text{N/mm}^2$, $D_0 = 10^4\ \text{sec}^{-1}$, $m = 100$ and $n = 1$. The height of the strip is chosen as $h = c/(5D_0)$, which means that an elastic shear wave whose speed is $c = (\mu/\rho)^{1/2}$ will propagate a distance of $5h$ during the time interval D_0^{-1}.

All the results presented in this paper are obtained with the spatial increments $\Delta x_1/h = \Delta x_2/h = 0.05$. With the time increment $c\,\Delta t/h = 0.01$, and the constant a in eqn (28) chosen such that $ah\Delta x_1/\Delta t = 1$, the steady-state solution is achieved, after 200 time steps. For the stability parameter γ in (29) we have chosen $\gamma = 1.1$. With these choices, the numerical scheme was found to be unconditionally stable and to provide solutions with a satisfactory accuracy. This is demonstrated in the sequel for the perfectly elastic strip, for which an analytical solution is known[9]. Finally, the applied displacement w_0 on the surface of the strip has been chosen as $w_0/h = 0.008$.

5.1 *Crack propagation in a perfectly elastic strip*

We present results for the case of steady propagation of a crack in a perfectly elastic strip to demonstrate the accuracy of the proposed method of solution by comparing the results with the closed form solution given in[9]. The excellent agreement suggests the potential usefulness of the method for the analysis of other elastic crack propagation problems.

Figure 2 shows comparisons between the numerical and analytical solutions for the shear stress σ_{32} and the displacement u_3, both versus x_1, in the plane of the crack $x_2 = 0$, for $v/c = 0$ and $v/c = 0.8$. The excellent agreement between the numerical and analytical solutions is evident. At the crack tip ($x_1 = x_2 = 0$) the analytical solution predicts, of course, an unbounded value for σ_{32}. It has been shown in Ref.[14] that the value of the shear stress obtained from the numerical solution at the closest grid point to the tip, i.e. at $(\Delta x_1, 0)$, can be employed to determine the stress intensity factor from the numerical results. Alternatively the method of calibrated crack-opening diaplacements[15] can be used for the determination of the stress intensity factor by employing the crack opening displacement at the point on the crack face which is nearest to the crack tip. Both approaches can be used here to determine K_3 for the present strip problem for different values of the crack-tip speed. A closed-form expression for K_3 is given in[10] as

$$K_3 = \mu w_0\,[2(1 - v^2/c^2)^{1/2}/h]^{1/2}. \tag{32}$$

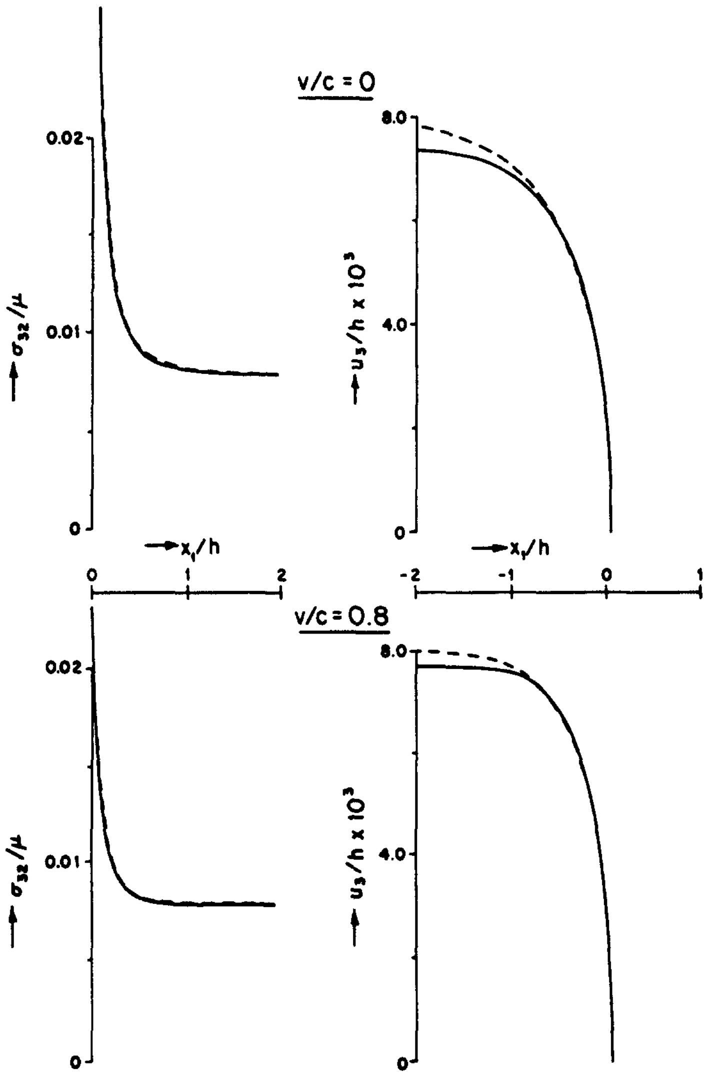

Fig. 2. Comparison of numerical (—) and analytical (---) solutions of σ_{32} and u_3 in the plane of the crack in a perfectly elastic strip for $v/c = 0$ and $v/c = 0.8$

5.2 *Crack propagation in a viscoplastic strip*

At some distance ahead of the crack tip in the loaded viscoplastic strip the inelastic field variables are given by (17) and the long time solution to (18)–(19). For our specific value of $w_0 = 0.008h$ we obtain $\sigma_{32}/\mu = 0.005$, $\epsilon_{32} = 0.004$, $\epsilon_{32}^{(p)} = 0.0016$, and $W_p/\mu = 0.2 \times 10^{-4}$. It may be noted that for a perfectly elastic strip we would obtain $\sigma_{32}/\mu = 0.008$.

Since the present unified theory contains no yield criterion and plastic strains are present at all stages of deformation, we consider an effective plastic strain whose rate is defined by

$$\dot{\bar{\epsilon}}_p = [2\, \dot{\epsilon}_{ij}^{(p)}\, \dot{\epsilon}_{ij}^{(p)}/3]^{1/2} \tag{33}$$

which reduces in the present problem to

$$\frac{\partial}{\partial x_1}\bar{\epsilon}_p = -2\Lambda(J_2/3)^{1/2}/v. \tag{34}$$

The state of deformation at a point can be specified by an offset rule according to which inelastic strains occur whenever $\bar{\epsilon}_p$ exceeds a certain value.

The numerical results have been used to study the effects of viscoplastic material behavior, crack tip speed and the effects of material inertia.

(a) *Effect of viscoplastic material behavior.* This effect can be exhibited by comparing the dynamic fields obtained for the elastic–viscoplastic strip and the perfectly elastic strip. In Fig. 3 we show the total strains ϵ_{32} in the plane of the crack, the shear stresses σ_{32} ahead of the crack tip and displacement on the crack-faces behind the crack tip, for $v/c = 0.1$ and $v/c = 0.8$. It is clearly noted that the effect of viscoplasticity is extremely pronounced especially at the lower velocity. The elastic crack-opening profile is steep, reflecting the square-root dependence on the distance from the tip. As was noted in Section 2 the stresses in the present constitutive model are necessarily bounded, but the total strains can be unbounded. It appears that the exact form of the strain singularity near the tip of the crack cannot be found by a simple asymptotic analysis. In the viscoplastic case the crack opens up more gradually, which suggests a weaker singularity in the total strain ϵ_{31} defined by (11). As the velocity increases σ_{31}, σ_{32} and ϵ_{31}, exhibit stronger intensities. This is also noticed in the graphs of ϵ_{32} if the magnitude of the jump from the value of residual plastic strain behing the crack tip (where $\epsilon_{32} = \epsilon_{32}^{(p)}$) to the total strain ahead of the crack tip is taken as a measure of the intensity of ϵ_{32} at the tip.

The weaker strain singularity which appears to occur in the inelastic problem (as compared to the stronger square root singularity in a perfectly elastic material) may be a primary reason for stable crack growth in a viscoplastic material. For quasi-static crack propagation under small scale yielding in an elastic perfectly-plastic material the strain singularity is known to be of a logarithimic type[16], and the crack-face displacement goes to zero like $x \log(-1/x)$ as $x \to -0$.

The observed tendency of the stresses and the total strains toward a higher intensity solution as the velocity of the crack tip in the rate-dependent material increases is consistent with the observation that the elastic strain range and the plastic flow stress increase with increasing strain rate.

(b) *Effect of crack-tip speed.* This effect, which is shown by comparing the various field variables for different crack-tip speeds in the viscoplastic material is directly related to the strain-rate dependence of the material. Figure 4 shows the total strains ϵ_{32}, the displacements u_3, the effective strains $\bar{\epsilon}_p$, and the plastic work W_p at $x_2 = 0$, for crack-tip velocities $v/c = 0.1$, 0.5 0.8. It is clearly seen that both the effective plastic strain and the plastic work behind the

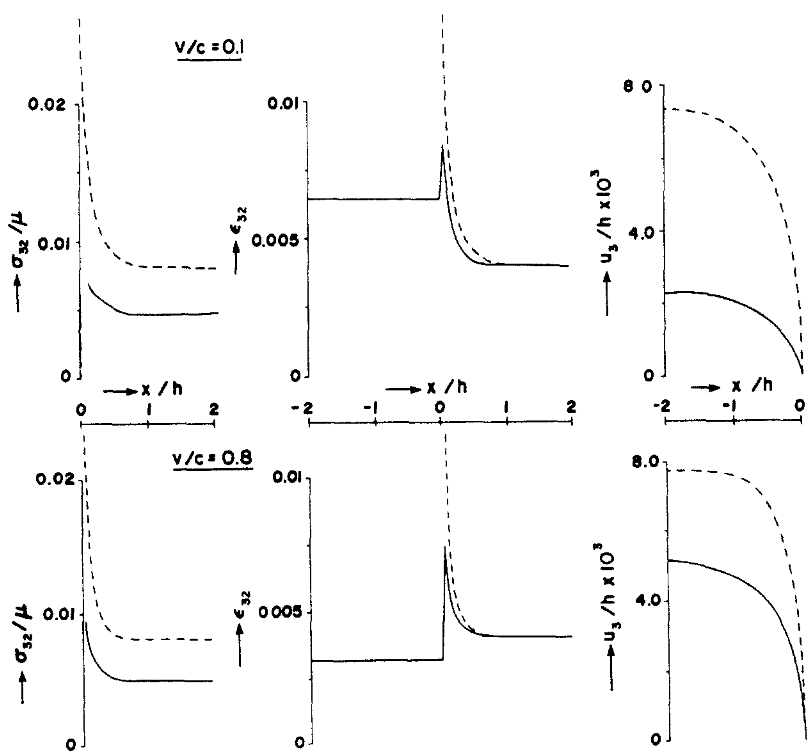

Fig. 3. Comparison of σ_{32}, ϵ_{32} and u_3 in the plane of a propagating crack in an elastic-viscoplastic strip (—) and a perfectly elastic strip (---), for $v/c = 0.1$ and $v/c = 0.8$.

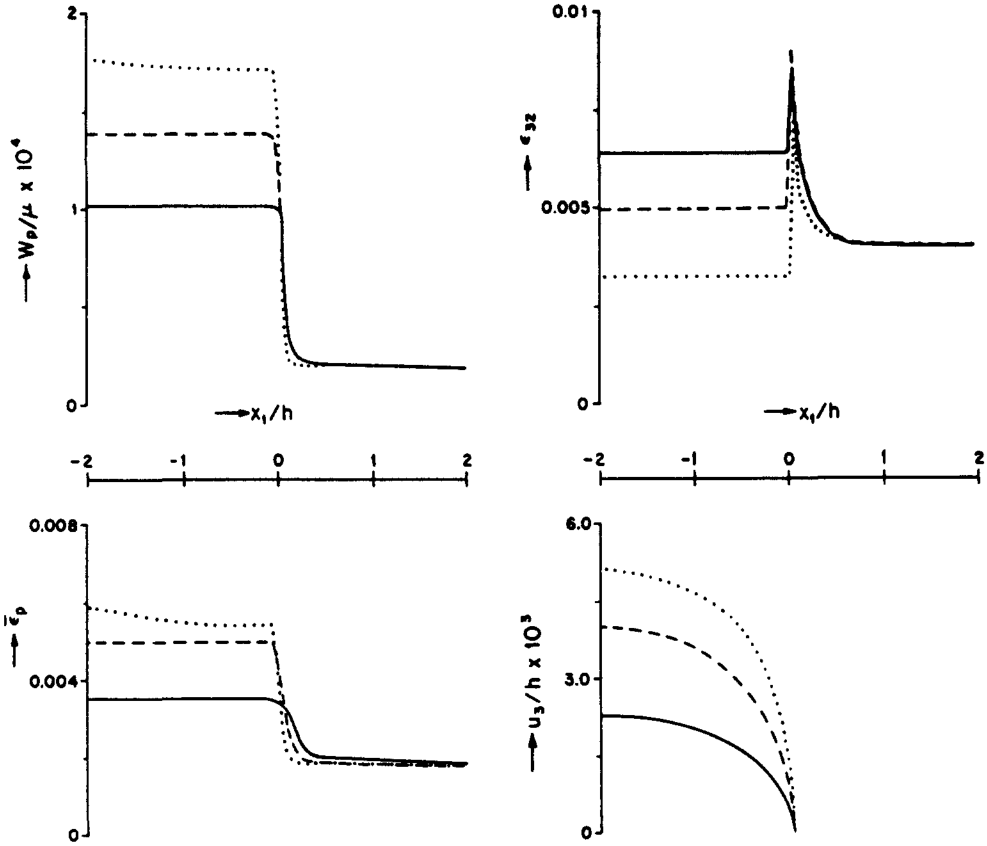

Fig. 4. Comparisons of W_p, ϵ_{32}, $\bar{\epsilon}_p$ and u_3 for a crack propagating in an elastic-viscoplastic strip, at different velocities: $v/c = 0.1$ (—), $v/c = 0.5$ (---) and $v/c = 0.8$ (· · ·).

crack tip increase with the velocity. Thus, the material in the wake experiences more severe plastic flow in the case of a rapidly moving crack. This effect is also exhibited in the plot of the plastic work behind the crack tip in the plane $x_2 = 0$. The increase of the effective plastic strains (or equivalently $2\epsilon_{32}^{(p)}/3^{1/2}$) in the active plastic region just ahead of the crack with increasing v can also be noted. In an elastic perfectly-plastic material with quasi-static crack growth the plastic strain contains a singularity of logarithmic type. It is interesting to note that for the lower and intermediate velocities $v/c = 0.1$ and 0.5, respectively, the wake is composed of residual plastic strains which remain constant after an unloading process has taken place. At higher velocity $v/c = 0.8$ on the other hand, there is a reversed loading effect back into the plastic region, which can be seen in the plots of $\bar{\epsilon}_p$ and the irreversible plastic work W_p. The occurrence of this secondary loading zone along the crack faces is caused by the high values of J_2 (or equivalently the effective stress $\bar{\sigma} = (3J_3)^{1/2}$), which are obtained when $v/c = 0.8$ and which cause an appreciable amount of plastic strain in addition to the already existing residual ones. This secondary zone was taken into account by Chitaley and McClintock[17], but it was found to be negligible in[3].

The extent of the active plastic zone ahead of the crack tip as a function of crack-tip speed can be obtained by a careful comparison of the plots of the effective plastic strain at different values of the crack-tip speed. This comparison shows that the length of the plastic zone decreases as the velocity of the crack tip increases. This appears to be consistent with the expectation that for a higher velocity (which gives rise to a higher rate of loading) the amount of plastic flow ahead of the crack tip should be less in this rate-dependent theory.

By arbitrarily adopting 0.2% as the value of effective plastic strain for a "plastic" region, we obtain the values shown in Table 1 for the dimensions of the region in which the material is in the "plastic" state, for the various crack-tip speeds. The Table shows also that the extent of the plastic zone in the x_2-direction is less than its extent directly ahead of the crack.

It should be interesting to compare our results for the rate-dependent hardening material

Table 1. The extent of the plastic regions in the strip for different crack-tip speeds. The last column corresponds to the quasi-static formulation in which the inertia term is omitted.

x_2/h	$v/c = 0.1$	$v/c = 0.5$	$v/c = 0.8$	$v/c = 0.8$ quasi-static
0	$x_1/h \leq 0.40$	$x_1/h \leq 0.25$	$x_1/h \leq 0.10$	$x_1/h \leq 0.25$
0.05	0.40	0.25	0.10	0.20
0.10	0.35	0.20	0.10	0.20
0.15	0.35	0.20	0.10	0.15
0.20	0.30	0.15	0.10	--
0.25	0.25	--	0.05	--
0.30	--	--	0.0	--
	--	--	--	--

with those obtained for an elastic perfectly-platic material with a steadily growing crack under quasi-static small-scale yielding conditions. In the case of a stationary crack the elastic plastic boundary is circular and extends along a distance $r_p = (K_3/\tau_0)^2/\pi$ ahead of the tip, see[16], where τ_0 is the yield stress in shear. For a growing crack the distance ahead of the tip is almost the same as that without growth[17]. It deceases, however, for linear strain-hardening and power-hardening materials[3].

Adopting an offset rule by which the yield stressd is determined at a permanent strain of 0.2%, we obtain for our material that the yield stress in shear is given approximately by $\tau_0/\mu = 0.008$. From (32) we have

$$(K_3/\tau_0)^2 = 2h(1 - v^2/c^2)^{1/2}\,[(w_0/h)/(\tau_0/\mu)]^2. \tag{35}$$

From (35) we obtain that the extent of the plastic region ahead of a quasi-static steadily growing crack in an elastic–plastic material is given by

$$r_p/h = (2/\pi)\,(1 - v^2/c^2)^{1/2}\,[(w_0/h)/(\tau_0/\mu)]^2 \tag{36}$$

which gives in the present situation the values $r_p/h = 0.63$, 0.55 asnd 0.38 for crack-tip speeds $v/c = 0.1$, 0.5 and 0.8, respectively. These values are seen to be higher than those appearing in Table 1 (at $x_2 = 0$) for our work-hardening material. They do, however, show the observed decrease of r_p with v. This decrease of the length of the active plastic zone ahead of the crack-tip due to work hardening is plausible. It was also obtained by Dean and Hutchinson[3] for quasi-static crack growth under small scale yielding conditions. As to the extent of the plastic region in the x_2 direction, the results of Chitaley and McClintock[17] give approximately the value of $r_p/3$ which is lower than the value of $r_p/2$ given by our results stated in the Table.

Fracture criteria for a growing crack in ductile materials have been based on critical values of one of the following near-tip fields: the plastic strain[17], the crack-opening displacement[2], the crack-opening angle[18], and the total strain in the plane of the crack[3].

In a plastic strain fracture criterion it is assumed that crack growth can initiate and continue if ahead of the crack-tip a critical amount of plastic strain accumulation is achieved:

$$\epsilon_{32}^{(p)} = \epsilon_c \quad \text{at } x_1 = x_c.$$

Our results show that this criterion can be easily applied (note that $\epsilon_{32}^{(p)} = \sqrt{(3)}\,\bar{\epsilon}_p/2$ for $x_1 > 0$ in the plane of the crack) and that it will exhibit a potentially increased instability of the crack at higher propagation speeds, see Fig. 4.

The criterion of a critical crack-opening displacement at a given distance behind the tip requires that

$$\delta = u_3(x_1, 0^+) - u_3(x_1, 0^{-1}) = \delta_c \quad \text{at } x_1 = -x_c.$$

This criterion can also easily be applied to our results.

The crack opening angle is defined as the angle enclosed by the crack faces behing the propagation crack tip. Since the free surface has a vertical tangent to the x_1-axis at the crack tip, the local crack opening angle is defined as the crack-opening displacement measured at a small fixed distance behind the crack tip divided by that distance. This criterion can be easily adopted and applied to our results.

The results exhibited in Fig. 4 for the total strain can, however, not conveniently be used for a fracture criterion.

(c) *Dynamic effects.*The effects that can be attributed to the inertia term in the equation of motion (14) can be studied by comparing the solution for the dynamic formulation in the rate-dependent material with the solution for the corresponding quasi-static formulation in which the inertia term is omitted. Such a comparison is given in Fig. 5 for a high crack-tip velocity, $v/c = 0.8$. This figure shows the plastic work, the total strains, the effective plastic strains and the displacements along the crack faces. It is clearly seen that a secondary plastic zone behind the crack tip is obtained in the dynamic problem. This zone does not exist in the quasi-static case, since the effective stress is not high enough to cause plastic flow. The considerable amount of plastic deformation which is present in the dynamic problem can also be seen in the plot of the plastic work, which shows a significant amount of dissipated energy. Also shown in Fig. 5 is the decrease of the slope of u_3 on the crack-faces for the quasi-static case indicating a weaker intensity and smaller crack opening angle. This shows that the quasi-static formulation would suggest more stable steady crack-growth than the dynamic formulation. The plot of $\bar{\epsilon}_p$ shows, however, a slight increase of the effective plastic strain at the tip of the crack in the quasi-static problem which, if $\bar{\epsilon}_p$ is adopted as a fracture criterion, would lead to the conclusion that the growth is somewhat less stable than for the corresponding dynamic formulation.

The extent of the "plastic" zone near the crack tip for the quasi-static problem is given in Table 1. It is seen that whereas its length in the crack propagation direction is considerably larger, its extent is smaller in the vertical direction.

For the intermediate crack-tip speed $v/c = 0.5$ the effect of the inertia term has decreased and it has completely disappeared for $v/c = 0.1$.

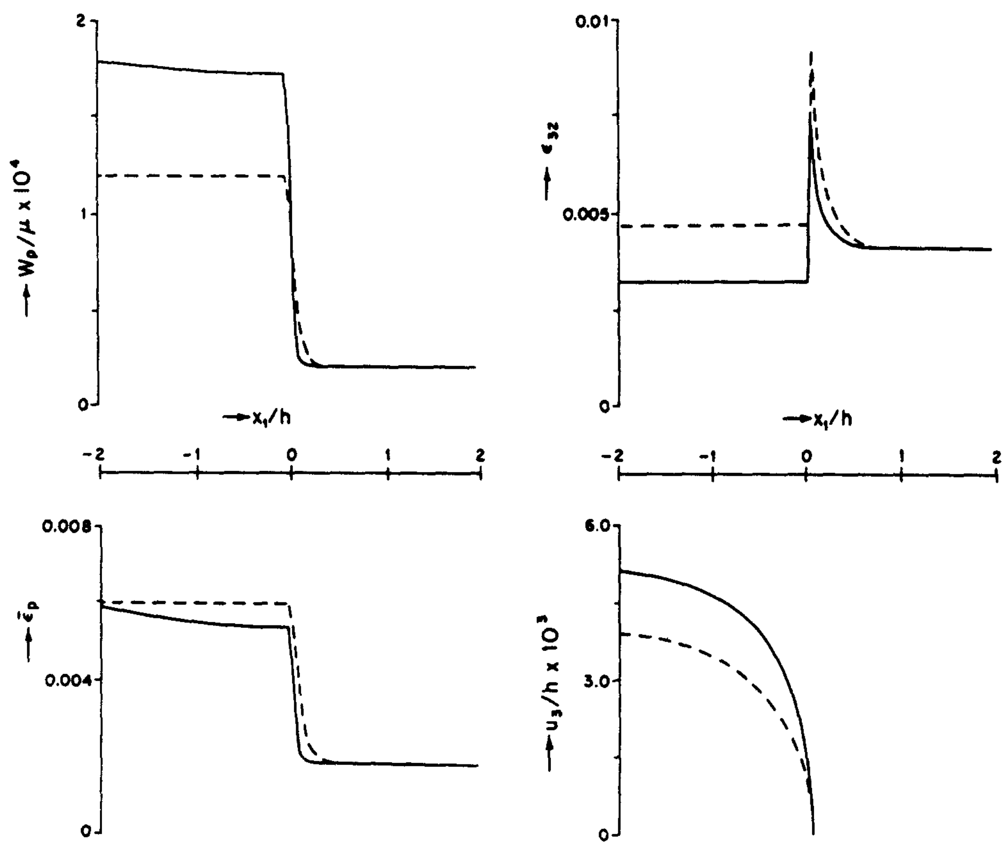

Fig. 5. Comparisons of W_p, ϵ_{32}, $\bar{\epsilon}_p$ and u_3 for the dynamical formulation (—) and the quasi-statical formulation (---), for $v/c = 0.8$ an an elastic-viscoplastic strip.

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